RECITATION 10

L'HÔPITAL'S RULE AND OPTIMIZATION

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Section 1. *Exercises*

Exercise 1

Evaluate $\lim_{x\to -1} \frac{1-x^2}{x+1}$ using l'Hôpital's rule.

Solution .:.

As $x \to -1$, the numerator approaches 0, as does the denominator. Hence we can apply l'Hôpital's rule, and get that

$$\lim_{x\to -1} \frac{1-x^2}{x+1} = \lim_{x\to -1} \frac{-2x}{1} = 2$$

Exercise 2

Can l'Hôpital's rule be applied to $\lim_{x\to 0} \frac{x}{\ln(x)}$?

Solution .:.

No: as x goes to 0, $\ln(x)$ goes to $-\infty$, so naïvely the limit wouldn't correspond to an indeterminate form: $0/-\infty$ isn't $\pm\infty/\infty$ nor $0/0$.

Exercise 3

Evaluate $\lim_{x\to\infty} e^x/x^3$ using l'Hôpital's rule.

Solution .:.

Applying l'Hôpital's rule three times, it follows that $\lim_{x\to\infty} e^x/x^3 = \lim_{x\to\infty} e^x/(3x^2) = \lim_{x\to\infty} e^x/6x = \lim_{x\to\infty} e^x/1 = \infty$.

Exercise 4

Minimize $x^2 + y^2$ assuming $xy = 4$.

Solution .:.

Write $f(x, y) = x^2 + y^2$. Since $y = 4/x$, we can regard f as a function of just x: $f(x) = x^2 + 16/x^2$. So now we just need to maximize this.

$f'(x) = 2x - 32/x^3$ and this is 0 iff $2x^4 = 32$, i.e. $x = \pm 2$, which give the critical points. $f''(x) = 2 + 96/x^4 > 0$ so that f is always concave up, and therefore $f(\pm 2) = 8$ is the minimum value of f.

Exercise 5

Maximize e^{xy} assuming $x + y = 2$ where $x \geq 0$ and $y \geq 0$.

Solution .:.

Note that $y = 1 - x$ so that we want to maximize $f(x) = e^{x(2-x)} = e^{2x-x^2}$. $f'(x) = (2-2x)e^{2x-x^2}$, which is 0 iff $x = 1$. For $x < 1$, $f'(x) > 0$ and for $x > 1$, $f'(x) < 0$. Hence $x = 1$ yields a maximum of $f(1) = e^{1 \cdot 1} = e$.

Exercise 6

Maximize the area of a right triangle assuming its hypotenuse is $\sqrt{2}$.

Solution .:.

With a hypotenuse $\sqrt{2}$, the legs of the triangle satisfy $x^2 + y^2 = 2$. The area of the right triangle is $xy/2$. Since $y > 0$, we have that $y = \sqrt{2 - x^2}$ and so we want to maximize $A(x) = xy(x)/2 = \frac{x\sqrt{2-x^2}}{2} = \frac{\sqrt{2x^2-x^4}}{2}$ (this holds because $x > 0$).

$$A'(x) = \frac{4x - 4x^3}{4\sqrt{2x^2 - x^4}} = \frac{4 - 4x^2}{4\sqrt{2x - x^2}} = \frac{1 - x^2}{\sqrt{2x - x^2}}.$$

This is 0 iff $1 = x^2$, i.e. $x = 1$. This is 0 iff $x = 0$ (which is impossible as we're assuming $x > 0$ as the leg length of a triangle) or $x = 2$ (which is impossible, as we're assuming $y > 0$ as the leg length of a triangle). Therefore $x = 1$ is the only critical point. One can check that for $0 < x < 1$, $A'(x) > 0$ while for $x > 1$, $A'(x) < 0$. So by the first-derivative test, $x = 1$ yields a maximum area of $A(1) = 1/2$.